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Partnership and Partition: A Case Study of Mathematical Exchange

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Résumé : Cela fait maintenant un peu plus de cent ans qu'a débuté le partenariat entre l'analyste de Cambridge G. H. Hardy et le génie indien des mathématiques Srinivasa Ramanujan, partenariat qui constitue l'une des plus célèbres collaborations de l'histoire des mathématiques. La manière dont Ramanujan est arrivé à Cambridge et l'accueil enthousiaste qu'il a reçu de la part de la communauté mathématique britannique sont aujourd'hui presque légendaires. Mais, dans le contexte de ce numéro, cet événement fournit une étude de cas intéressante d'échange mathématique. Cet article examine un résultat particulier dû au partenariat créatif entre Hardy et Ramanujan, leur article de 1918 sur les partitions, et montre que l'échange que leur travail sur cet article a provoqué a été en partie facilité par leur appartenance à la première société savante pour la promotion de la recherche mathématique en Grande-Bretagne : la London Mathematical Society.

Abstract: It is now just over one hundred years since the beginning of the mathematical partnership between the Cambridge analyst G. H. Hardy and the Indian mathematical genius Srinivasa Ramanujan, one of the most celebrated collaborations in the history of mathematics. Indeed, the story of how Ramanujan was brought from India to Cambridge and feted by the British mathematical establishment now borders on legendary. But, in the context of this collection of articles, it provides an interesting case study of mathematical exchange. This paper considers one particular product of the Hardy-Ramanujan creative partnership : their 1918 paper on partitions, and argues that the exchange of ideas prompted by their work on this paper was facilitated in part by their membership of the premier learned body in Britain for the advancement of mathematical research : the London Mathematical Society.

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1 Introduction: The London Mathematical Society

The London Mathematical Society (or LMS) was founded in 1865 as little more than a student club at University College London [Rice, Wilson *et al.* 1995]. However, under the inaugural presidency of Augustus De Morgan, it quickly developed into what was, in essence, the national British learned society for mathematics. With its stated aim of "the cultivation of pure mathematics and their most immediate applications" [De Morgan 1866, 2], the LMS grew to comprise over one hundred members in two years, from famous mathematicians like Cayley and Sylvester, to schoolteachers, civil servants, lawyers and clergymen. In the words of former LMS President, Harold Davenport, the Society "brought together not only the leading mathematicians of the country but also others who were pursuing mathematical research in isolation, while earning a living in some profession" [Davenport 1966, 2]. Another erstwhile President J.W.L. Glaisher added that the Society "drew from their seclusion not only workers but others who had previously had no means of showing their interest in mathematical progress [...] who otherwise would not easily have had opportunities of becoming personally acquainted with one another" [Glaisher 1926, 55]. The Society thus played a key role in bringing mathematicians from diverse backgrounds into contact with one another and deserves much of the credit for the improvement of mathematical communication in Britain towards the end of the 19th century.

Right from the outset, the Society's principal function was to hold monthly meetings—for the presentation and exchange of mathematical research—and to publish and disseminate these papers in a refereed journal. In contrast with other British scientific societies at the time, the publication policy of the LMS was unusually strict. Whereas organizations such as the Royal Society or the Royal Astronomical Society usually only required that papers submitted by non-members be adjudicated by a single referee, at the LMS no paper was published until written reports had been received from *two* independent referees,[1] and even then, publication was decided by a secret ballot of LMS Council members. Glaisher, who served on the Council from 1872 to 1907, later recalled:[2]

> In the [London] Mathematical Society every paper was invariably considered by two referees, who sent in written reports which were

1. The first reference to this refereeing procedure appears in the LMS Council Minute Book for 19 March 1866 [LMSCM, Vol. I, 1866, f.2]; see also [Heard 2004, 76–77]. Sadly, little evidence exists today of this refereeing process because the referees' reports were not preserved by the LMS.

2. The article in which Glaisher's recollections were contained [Glaisher 1926] was originally delivered as an address at a meeting on 11 June 1925 to commemorate the Society's sixtieth anniversary. It was subsequently published in the inaugural issue of the Society's new publication, the *Journal of the London Mathematical Society*.

> read to the Council; and when the reports differed the paper was sent to a third referee. Every paper was balloted for, to decide whether it should be printed [...]. At the [Royal] Astronomical Society, on the contrary, it was rarely that a paper was refereed, and a verbal report from a single referee was generally accepted [...]. The strict procedure in the Mathematical Society with regard to the treatment of papers was in operation when I became a member of the Council of the Mathematical Society, and it was then quite established, and I presume must have existed almost from the foundation of the Society. It underwent no modification while I was on the Council, and it has continued, I believe, to the present time. [Glaisher 1926, 60]

This rigorous procedure, while burdensome, appears to have been successful since Glaisher also reported that:

> In no case in the writer's experience was there any bias; nor was any distinction made in favour of distinguished mathematicians or on personal grounds. All papers were adjudicated upon by exactly the same procedure and with the same impartiality.[3] [Glaisher 1914, liii]

Indeed, there is evidence that even distinguished mathematicians had papers rejected from time to time. For example, entries from the LMS Council minute books note the rejection of submissions by Karl Pearson [LMSCM, vol. II, 1885, f.72], Grace Chisholm Young [LMSCM, vol. V, 1903, f.105] and Louis Mordell [LMSCM, vol. V, 1913, f.214]. Reasons for rejections varied, but not all were due to lack of quality, as the following anecdote suggests:

> J. J. Sylvester [once] sent a paper to the London Mathematical Society. His covering letter explained, as usual, that this was the most important result in the subject for 20 years. The Secretary replied that he agreed entirely with Sylvester's opinion of the paper; but Sylvester had actually published the result in the L.M.S. five years before. [Bollobás 1986, 148]

To further promote the Society's reputation and the work of its members, the LMS quickly established exchange agreements, whereby its *Proceedings* were sent to other learned societies in exchange for copies of their periodicals. Reciprocal arrangements were made with the Royal Society, the Cambridge Philosophical Society, the Philosophical Society of Manchester, the Royal Irish Academy, Royal Society of Edinburgh, the Académie des sciences, Accademia dei Lincei, National Academy of Sciences (Washington DC),

3. As a prominent member of the LMS Council for over three decades, Glaisher was a regular participant in the early activities of the LMS, including its refereeing process. His first-hand accounts of the early years of the LMS are thus of significant value to the historian.

as well as journals, such as Crelle's *Journal*, Liouville's *Journal, Annali di matematica, Mathematische Annalen, Bulletin des sciences mathématiques et astronomiques, American Journal of Mathematics*, and the *Rendiconti del Circolo Matematico di Palermo.*

This notion of intellectual exchange extended to its membership, since the LMS also shared members with other societies, both at home and abroad. Indeed, as early as April 1866, the LMS Council passed a resolution:

> That persons being neither British subjects nor residing in Her Majesty's dominions shall be selected from among mathematicians of the greatest eminence for Honorary Membership. [LMSCM, vol. I, 1866, f.3]

The first of these was the French geometer Michel Chasles, who had initially applied for ordinary membership in March 1867 [LMSCM, vol. I, 1867, f.17], [Collingwood 1966, 584], and was elected an honorary foreign member of the Society the following month. Subsequent honorary foreign members included: Eugenio Beltrami, Enrico Betti, Francesco Brioschi, Georg Cantor, Rudolf Clebsch, Luigi Cremona, Jean-Gaston Darboux, Josiah Willard Gibbs, Paul Gordan, Charles Hermite, Otto Hesse, David Hilbert, Felix Klein, Leopold Kronecker, Sophus Lie, Gösta Mittag-Leffler, Émile Picard, Henri Poincaré, Hermann Schwarz and H.G. Zeuthen.

The establishment of a learned body devoted entirely to mathematics must have struck these foreign members as a somewhat novel idea, since the LMS was one of the first such societies in existence, and was certainly the first to exert a major influence on other mathematical communities. As Chasles famously reminded his countrymen in 1870:

> [...] a mathematical society was founded in London in 1865 with a membership of one hundred; [...] a society whose *Proceedings,* like those of the Royal Society of London [...], publishes abstracts, more or less extended, of many papers. Is not this fact [the existence of the *Proceedings*], which we applaud, an indication of future superiority in mathematical culture that should worry us?[4] [Chasles 1870, 379]

One result of Chasles' influential comments was the subsequent formation of the Société mathématique de France [Gispert 2015], as reported in the LMS *Proceedings* in 1872:

4. "[...] il s'est formé à Londres, en 1865, une Société mathématique d'une centaine de membres, et le nombre s'en accroît encore; société dont les *Proceedings*, à l'instar de la Société royale de Londres [...], font connaître les travaux par des analyses plus ou moins étendues. Ce fait [l'existence des *Proceedings*] auquel nous applaudissons n'est-il pas dans la culture des Mathématiques un élément de supériorité future qui doit nous préoccuper?"

> A Mathematical Society of Paris has been founded [. . .], having for its object to encourage mathematical studies, and increase mathematical knowledge, and to form a bond of union of those interested in the mathematical sciences. [LMS Proceedings 1872, 4: 419]

The Société mathématique de France was not the only mathematical society established in emulation of the example set by the LMS. The LMS also served as a model for other subsequently founded mathematical societies, such as the Circolo Matematico di Palermo and the American Mathematical Society [Parshall 1996, 293–294], the births of which further illustrated the growth of the professionalization of mathematics in Europe and North America at this time. By the turn of the century, the importance of international communication and exchange as a means of furthering their subject was universally recognized by mathematicians. As Karen Parshall notes:

> By the late nineteenth century, to be a mathematician meant the same thing internationally: namely, to produce and to share the results of original research with like-minded members of an extended community of mathematical scholars both at home and abroad. [Parshall 1996, 294]

2 The *sociabilités* of the LMS

As a new and developing institution, the LMS was a site with a variety of *sociabilités* in its early years, and as the Society changed and grew, its *sociabilités* evolved similarly. The character of the LMS went through several (sometimes overlapping) stages in its formative years. At the very beginning, in 1865, it was simply a student club at University College London with 26 out of its initial 27 members having either a past or present connection with the College [Rice, Wilson *et al.* 1995, 408]. Its initial exchanges, therefore, were limited to other student organisations within University College, such as the debating and literary societies, with whom it had members in common. But as the LMS quickly grew and attracted members from outside University College, it soon lost its status as a college society, as is documented in the *University College Gazette*:

> Among University College Societies, the Mathematical Society [. . .] under Professor De Morgan as President, and Dr. T. Hirst as Vice-President, should not be forgotten. This Society soon attracted the notice of some of the foremost mathematicians of the country, and from being a University College Society it developed into the Mathematical Society of London and removed from the College to quarters of its own in 1867 [. . .]. [Notes 1888, 90]

The LMS was now one of several London-based learned societies. As such, it developed cordial relationships with its peer institutions, not only via the exchange of publications and reciprocity of membership, but also by the fact that due to lack of finances it was forced to share its accommodation with some of them. For example, from its move from University College until 1870, the LMS held its meetings in rooms loaned by the Chemical Society; then, from 1870 until 1916, it met in a building owned by the British Association for the Advancement of Science, in rooms which were let to the Royal Asiatic Society [Rice & Wilson 1998, 190]. The contacts of the LMS thus now concerned fellow organisations in the literary and scientific milieu of the capital.

By this point, although firmly based in London, the LMS had also become a national scientific society, comprising members not just from London but from all over the United Kingdom. The LMS now shared members with prestigious national bodies such as the Royal Society, the Institute of Actuaries and the Royal Astronomical Society, and in several cases prominent positions in other societies were held by LMS members.[5] Before long, the contacts of the LMS had spread across the English Channel, not simply via the election of foreign members and the exchange of journals, but also via the attendance and participation of overseas mathematicians at LMS meetings. Foreign attendees in this period included Camille Jordan, Ferdinand Lindemann and Gösta Mittag-Leffler [Rice & Wilson 1998, 197, 207], while several mathematicians from abroad who did not attend in person (such as Charles Hermite, David Hilbert and Felix Klein) submitted papers for publication in the LMS *Proceedings*. By the end of the 19th century, via its mutual relationships with burgeoning national societies overseas, the contacts of the LMS extended as far afield as the United States of America.

It was not just the relationship of the LMS with external bodies that changed over time. The composition of LMS membership changed significantly during its first fifty years, comprising mostly amateur mathematicians in 1865 but becoming markedly more professional in nature by 1915. Consequently, the exchanges between of many of its members were very different in 1915 from what they had been fifty years earlier. At its inception, almost all LMS members were former or current students of University College London, but only four of the original 27 members were (or would become) professional mathematicians. The vast majority went on to careers in school education, business or, in several cases, law [Rice, Wilson *et al.* 1995, 407–409].

Indeed, the early membership of the LMS is dominated by those with a strong amateur interest in mathematics, but whose employment lay in a different field entirely. These included: the Reverend Robert Harley, a church minister and self-taught mathematician, whose principal area of research was Boolean symbolic logic [Collingwood 1966, 583]; Sir James Cockle, a promi-

5. For example, William Spottiswoode, Lord Kelvin and Lord Rayleigh were all Presidents of the Royal Society, while Arthur Cayley, J. W. L. Glaisher and Percy MacMahon served as Presidents of the Royal Astronomical Society.

nent lawyer and onetime Chief Justice of Queensland, Australia, whose interests included algebra and hypercomplex numbers [Collingwood 1966, 583]; and Samuel Roberts, a trained solicitor who spent much of his life writing papers on pure mathematical subjects, but without ever holding an academic position [Glaisher 1926, 53]. Workers from a variety of non-mathematical professions were thus affiliated to the LMS by their membership and formed a large and distinct part of the Society's early clientele. Amateur mathematicians were clearly a vital and very welcome constituent of the LMS in its early years.[6]

Another population that was well represented within the LMS was schoolteachers, to whom the LMS provided a valuable source of mathematical communication and exchange. Indeed, it has been said that had it not been for the LMS, some of these scholars would have had no opportunity to engage with their fellow mathematical researchers. One such member was Thomas Cotterill, a Cambridge graduate and London schoolmaster, who "seems to have been quite unknown to his contemporaries, but he surprised all who met him at the Society by his knowledge of so many branches of mathematics and their recent developments" [Glaisher 1926, 55]. Another teacher, J. J. Walker, was apparently "very quiet and unassuming in manner, and but for the Society he would probably have remained personally unknown to most of his fellow workers" [Glaisher 1926, 55–56]. The LMS thus provided a venue whereby those who would previously have been forced to pursue their mathematical research alone, or at best via correspondence, were now given the ability to share their ideas in person.

But despite early active involvement from many members who conducted mathematical research in their spare time, by the opening years of the 20th century, it was contributions from university-based professional research mathematicians which dominated the output of the LMS. Although this particular group (including mathematicians such as Cayley, Sylvester and Smith) had been present virtually from the very beginning, it was only in the opening decades of the twentieth century that a true research ethos really came into being at British universities [Heard 2004]. The consequent expansion of university mathematics departments produced more research-minded mathematicians, many of whom subsequently joined the LMS, which was by this time the *de facto* British learned society for mathematical research. By the early twentieth century, the LMS *Proceedings* was reflecting this growing professionalization, as more and more LMS members viewed mathematics as a vocation rather than a pastime.[7]

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6. It should, however, be noted that not all contributions from amateur mathematicians were well received by the LMS. At a meeting on 9 November 1871, it was resolved that no papers claiming to demonstrate the squaring of the circle would be accepted [Collingwood 1966, 584].

7. For more on the notion of professionalization of mathematicians in Britain during this period, see [Heard 2004].

By the outbreak of the First World War, then, the majority of contributions to LMS meetings and publications were by university-trained research mathematicians. But this dominance was not exclusive. The LMS was still a venue in which professional and amateur could meet and exchange mathematical ideas, and it is to an example of just such a mathematical exchange that we now turn. This exchange is of interest, not only because of the facilitating role played by the LMS, but also because its protagonist was a mathematician engaged in *sociabilités* concerning mathematical researchers and who, more than any other, would influence the Society's development over the next few decades into a body entirely for the professional mathematician [Rice & Wilson 2003]. His name was G. H. Hardy.

3 Hardy, Ramanujan and partitions

Godfrey Harold Hardy was one of the finest British mathematicians of the twentieth century, with publications covering many areas of pure mathematics: Fourier series, Diophantine analysis, the summation of divergent series, the Riemann zeta function and the distribution of primes, much of this done in collaboration with his Cambridge colleague, J. E. Littlewood. He spent the vast majority of his professional life as a fellow of Trinity College, Cambridge, having arrived there as a student in 1896 and, with the exception of an eleven-year period in Oxford from 1920-1931, he lived there until his death in 1947 [Titchmarsh 1950]. As a mathematician Hardy was, first and foremost, an analyst, having discovered a love of function theory by reading Jordan's *Cours d'analyse* as an undergraduate [Hardy 1940a, 87]. Consequently, in the style of continental contemporaries such as Borel, Lebesgue and Landau, his mathematics was formal, proof-orientated, and highly rigorous.

Srinivasa Ramanujan, on the other hand, had little or no university education and minimal formal training in mathematics. Born in southern India in 1887, he was gifted with outstanding formulaic abilities in mathematics as well as remarkable mathematical intuition [Kanigel 1991]. His primary source of higher mathematical education appears to have been a book entitled *A Synopsis of Elementary Results in Pure Mathematics* by G. S. Carr [Carr 1886], which was a 935-page compendium of mathematical definitions, formulae and methods, presented in a dry and concise manner with little or no explanations or justifications. This was to exercise a profound influence on his style of mathematical presentation, as one Indian contemporary noted:

> Mr. Ramanujan's methods were so terse and novel and his presentation was so lacking in clearness and precision, that the ordinary reader, unaccustomed to such intellectual gymnastics, could hardly follow him. [Seshu Iyer 1920, 83]

Another consequence of his lack of formal mathematical education was that whole areas of modern mathematics were unknown to him. For example, despite an extensive knowledge of elliptic and modular functions, he was totally ignorant of complex function theory [Hardy 1921, lii]. Perhaps also due to his reliance on Carr's book, he had no real interest in rigorous justifications and "the clear-cut idea of what is *meant* by a proof [...] he perhaps did not possess at all" [Littlewood 1929, 426–427]. But despite this, by the age of 25, he had produced a startling array of beautiful and original mathematics.

In January 1913, Hardy received the now famous introductory letter from Ramanujan containing over fifty of his unproved results. Hardy later wrote of his amazement at the power of his correspondent's intellect:

> [...] he had never seen a French or German [mathematics] book; his knowledge even of English was insufficient to enable him to qualify for a degree. It is sufficiently marvelous that he should have even dreamt of problems such as these, problems which it has taken the finest mathematicians in Europe a hundred years to solve [...]. [Hardy 1921, xlv]

Arrangements were quickly made for Ramanujan to come to Cambridge in 1914 to work with Hardy. Although he was not elected a member of the LMS until 1917, Ramanujan quickly began publishing his research in British journals, including a lengthy memoir on "Highly composite numbers", which appeared in the LMS *Proceedings* in 1915 [Ramanujan 1915]. During their five-year partnership Hardy and Ramanujan co-authored several notable papers, but it was their penultimate publication, on the theory of partitions, that provides perhaps the finest example of mathematical exchange within the context of the LMS in the early twentieth century.

The subject of partitions lies on the border between number theory and combinatorics, consisting, initially at least, of problems which are easy to state and to understand, but which are remarkably difficult to solve. If $p(n)$ represents the number of ways that a positive integer n can be written as a sum of positive integers where the order of addition is irrelevant, then it is clear for example that, the number $n = 4$ can be written in five different ways and

$$\begin{aligned} 4 &= 3+1 \\ &= 2+2 \\ &= 2+1+1 \\ &= 1+1+1+1, \end{aligned}$$

that therefore, $p(4) = 5$. But the question quickly becomes more difficult. For example, $p(15) = 176$, and $p(34) = 12,310$. What then is $p(100)$? Or $p(200)$? It is clear that some theory is needed in order to provide a satisfactory answer.

The theory of partitions began in the 1740s with Euler, who proved some fundamental results using generating functions [Euler 1748, 1: 253–275], such as:

$$1+\sum_{n=1}^{\infty} p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\dots}$$

In the 1840s and 1850s, a number of English mathematicians, such as Augustus De Morgan, Henry Warburton, John Herschel, Thomas Kirkman and Arthur Cayley—[De Morgan 1845], [Warburton 1849], [Herschel 1850], [Kirkman 1855], [Cayley 1856]—worked on partitions using methods of finite differences. In 1857, James Joseph Sylvester [Sylvester 1857] pioneered the use of Cauchy's residue theorem in studying the restricted partition function $p_r(n)$,[8] where

$$1+\sum_{n=1}^{\infty} p_r(n)x^n = \frac{1}{(1-x)(1-x^2)\dots(1-x^r)}.$$

Sylvester later, assisted by Fabian Franklin [Sylvester & Franklin 1882], introduced the use of combinatorial and graph-theoretic methods to the subject, improving on many of Euler's earlier results. Sylvester and Cayley's work on partitions was continued in the algebraic style by Cayley's protégé J. W. L. Glaisher [Glaisher 1883], and in the combinatorial style by former British army officer, Major Percy MacMahon. All were LMS members who published regularly in the LMS *Proceedings*. But by the publication of MacMahon's magnum opus *Combinatory Analysis* in 1915, the subject of partitions was seriously neglected by British mathematicians. Indeed, as MacMahon had pointed out twenty years earlier in his valedictory presidential address to the LMS:

> The theory [of partitions] requires further elucidation and development, and it is to be hoped that workers in our science will now, after a period of forty years, give it some attention. [MacMahon 1896, 22]

That attention was duly given in Hardy & Ramanujan's monumental paper "Asymptotic formulae in combinatory analysis", presented at an LMS meeting on 18 January 1917 and published in the *LMS Proceedings* the following year [Hardy & Ramanujan 1918].[9] The paper had its genesis in one of Ramanujan's unproved (and, as it turned out, false) conjectures concerning partitions of natural numbers. In his original letter to Hardy, Ramanujan had claimed [Berndt & Rankin 1995, 28], that the coefficicient of x^n in $(1-2x+2x^4-2x^9+2x^{16}-\dots)^{-1}$ is the integer nearest to

$$\frac{1}{4n}\left(\cosh \pi\sqrt{n} - \frac{\sinh \pi\sqrt{n}}{\pi\sqrt{n}}\right),$$

8. This function represents the number of partitions of n into parts none of which exceed r.

9. A preliminary account of their results ("Une formule asymptotique pour le nombre des partitions de n") appeared in the *Comptes Rendus* of the Académie des sciences on 2 January 1917.

a claim which, although incorrect,[10] according to Hardy was "one of the most fruitful he ever made, since it ended by leading us to all our joint work on partitions" [Hardy 1940b, 9].

Their first step towards finding an expression for $p(n)$ had been their first joint paper for the LMS [Hardy & Ramanujan 1917, 130], in which Hardy & Ramanujan proved that

$$p(n) \sim e^{\pi\sqrt{2n/3}}.$$

Hardy then expressed Euler's function

$$f(x) = 1 + \sum_{n=1}^{\infty} p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}$$

essentially as the reciprocal of the Dedekind eta-function,

$$\eta(x) = x^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-x^n) = \frac{x^{\frac{1}{24}}}{f(x)}$$

where $x = e^{2\pi i\tau}$ and $Im(\tau) > 0$, and invoked Cauchy's residue theorem to represent $p(n)$ as the integral

$$p(n) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(x)}{x^{n+1}}dx$$

around a closed path Γ entirely within the unit circle enclosing the origin.[11] A close study of the behavior of this integral then gave the improved formula

$$p(n) \sim \frac{1}{4n\sqrt{3}}e^{\pi\sqrt{2n/3}}.$$

But Ramanujan now insisted that a more accurate approximation was possible. This necessitated the construction of an auxiliary function which, when appended to $f(x)$, ensured a far more precise estimate. By means of a clever transformation due to the Finnish mathematician Ernst Lindelöf [Lindelöf 1905, 111], this new function was converted into an integral and Cauchy's theorem applied again. The result was yet greater accuracy—but Ramanujan was still not satisfied: he was convinced that a more precise formula still existed. In "an extraordinary stroke of formal genius" [Littlewood 1929, 427], in place of the n's in the exponential part of the function, he substituted $(n - \frac{1}{24})$ and replaced the function with its first derivative. Quite what led Ramanujan to make such an unexpected (and ingenious) refinement has never been fully explained. For Littlewood, "there is, indeed, a touch of real mystery" [Littlewood

10. A correct version of this statement can be found in [Hardy & Ramanujan 1918, 109–110].

11. This application of techniques from complex analysis became known as the 'Circle Method' and was to become a highly useful tool in analytic number theory.

1929, 427] about the inspiration for this step, while according to Hardy this result, like every other by Ramanujan, was "arrived at by a process of mingled argument, intuition, and induction, of which he was entirely unable to give any coherent account" [Hardy 1921, lii]. But the end result was a triumph: Hardy & Ramanujan had proved that $p(n)$, the number of unrestricted partitions of the natural number n, is given by the integer nearest to

$$\frac{1}{2\pi\sqrt{2}}\sum_{k=1}^{v} A_k(n)\sqrt{k}\,\frac{d}{dn}\left(\frac{e^{c\lambda_n/k}}{\lambda_n}\right), \tag{1}$$

where $c = \pi\sqrt{2/3}$, $\lambda_n = \sqrt{n - 1/24}$, and

$$A_k(n) = \sum_{\substack{h=1\\(h,k)=1}}^{k} \omega_{h,k} e^{-2nh\pi i/k}$$

with $\omega_{h,k}$ being a particular $24k$-th root of unity.

Littlewood later described this result as "a very astonishing theorem" [Littlewood 1929, 427] and it is not difficult to see why. Hardy & Ramanujan's paper showed a real mixture of mathematical ideas and influences: a true mathematical exchange. Fundamentally, it was a fusion of Ramanujan's dazzling powers of formulaic intuition with Hardy's mastery of the tools of analytic function theory. To Hardy was due the incorporation of material derived from Tannery & Molk's four-volume *Théorie des fonctions elliptiques* [Tannery & Molk 1896, 31–32, 104–106, 113, 265, 267], analytic techniques from an earlier 1914 paper by Hardy & Littlewood [Hardy & Littlewood 1914], and a complex integral transformation formula arising from Ernst Lindelöf's application of results by Mittag-Leffler to the study of asymptotic series [Lindelöf 1905, 111]. But without the intuitive genius of Ramanujan, the theorem could not have been formulated at all. From his initial conjecture to his instinctive conviction that a closer asymptotic expression *must* exist, Ramanujan's formulaic virtuosity was an absolutely essential ingredient in this collaboration. It is practically certain that without Ramanujan, Hardy would never have formulated such an astonishing result; and without Hardy, Ramanujan would never have been able to prove it. "We owe the theorem", said Littlewood, "to a singularly happy collaboration of two men, of quite unlike gifts, in which each contributed the best, most characteristic, and most fortunate work that was in him" [Littlewood 1929, 427–428]. And yet this paper could not have existed in its final form without the LMS. For there was another crucial contributor to the project, whose exchanges with Hardy were fostered by their respective roles at the LMS: this contributor was Major Percy MacMahon.

MacMahon was a retired British army officer, who had previously taught physics at the Royal Artillery College in Woolwich. Born in 1854, his early research centered on the algebraic subjects favored by many British mathematicians of the late nineteenth century, particularly invariant theory, where his prodigious skill in computation and symbolic manipulation was effectively employed [Baker 1930]. By 1916, then in his sixties and working at the Board of Trade in London, MacMahon was a mathematician of the previous generation, more at home with the labour-intensive computations of Cayley and Sylvester's invariant theory than the subtle complexities of Hardy and Littlewood's analysis. But both he and Hardy had been attending the same LMS meetings for well over a decade, and had served together on the LMS Council since 1905. So when Hardy and Ramanujan needed a formidable human calculator to verify the accuracy of their partition formula, it was MacMahon to whom they turned.

Indeed, as Hardy later recalled, "[a]t this point we might have stopped had it not been for Major MacMahon's love of calculation" [Hardy 1940b, 119]. At their request, MacMahon single-handedly calculated every value of $p(n)$ for $n = 1$ to 200, using the recurrence relation

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - p(n-15) + \ldots = 0.$$

MacMahon's computations provided crucial numerical corroboration of the validity of Hardy & Ramanujan's formula:

> We expected a good result, with an error of perhaps one or two figures, but we had never dared to hope for such a result as we found. [Hardy 1940b, 119]

For example, the first six terms of their formula gave $p(100) = 190,569,291.996$, whereas MacMahon's precise number was $p(100) = 190,569,292$. Most astonishingly, the first eight terms of their formula gave $p(200) = 3,972,999,029,388.004$, "and Major MacMahon's subsequent calculations showed that $p(200)$ is, in fact, $3,972,999,029,388$" [Hardy & Ramanujan 1918, 84]. It was at this point, as Hardy later wrote, that "[w]e were inevitably led to ask whether the formula could not be used to calculate $p(n)$ *exactly* for any large n" [Hardy 1940b, 119]. In order to do this, in their formula (1), Hardy and Ramanujan now made υ a function of n, specifically, the integral part of $\alpha\sqrt{n}$, where α is an arbitrary positive constant. "This", according to Littlewood, "*was* a great step" and as a result, "the complete theorem thus emerged" [Littlewood 1929, 427]. In December 1916, as the paper neared completion, Ramanujan wrote to Hardy that "Major MacMahon was kind enough to send me a copy of the 200 numbers"—referring to the computed values of $p(n)$—and reported excitedly: "The approximation gives the exact number" [Berndt & Rankin 1995, 141]. The result was "one of the rare formulae which are both asymptotic and exact; it tells us all we want to know about the order and approximate form of $p(n)$, and it appears also to be adapted for exact calculation" [Hardy 1940b, 119]. It is perhaps not surprising that, as Hardy & Ramanujan acknowledged in their paper:

> To Major MacMahon in particular we owe many thanks for the amount of trouble he has taken over very tedious calculations. It

> is certain that, without the encouragement given by the results of these calculations, we should never have attempted to prove theoretical results at all comparable in precision with those which we have enunciated. [Hardy & Ramanujan 1918, 85–86]

But the exchange of mathematical ideas did not only travel in one direction. It was every bit a dialogue, since Hardy and Ramanujan's ideas also stimulated MacMahon's own research on the subject of partitions [MacMahon 1926]. And the exchange did not stop there. Thanks to its publication in widely read journals, Hardy & Ramanujan's work directly influenced subsequent research by fellow LMS members such as the analyst George N. Watson, who published nearly thirty papers inspired by Ramanujan's work, the number-theorist Louis Mordell [Mordell 1922], and even mathematicians further afield such as the American computational number-theorist Derrick Lehmer [Lehmer 1936, 1937], the Indian mathematician Hansraj Gupta [Gupta 1935, 1937] and the German analytic number-theorist Hans Rademacher, who in 1937, published an *exact* formula for the partition function for all n [Rademacher 1937]. All of these papers were published in LMS journals.

4 Conclusion

The story behind Hardy and Ramanujan's asymptotic formula for $p(n)$ is illustrative of the kind of mathematical exchange facilitated by bodies like the LMS, although at first sight a collaboration between mathematicians such as Hardy and MacMahon appears unlikely. Socially, academically and politically, they were almost complete opposites, and even mathematically, their styles and approaches contrasted dramatically. Yet, as Hardy and Ramanujan acknowledged, their paper could not have reached its final form without MacMahon's vital input—and this input was facilitated in part by the LMS. In the years before its creation, if a Cambridge mathematician like Hardy had wanted to consult a London-based army man like MacMahon, he would have had to introduce himself by letter and their exchange would have continued almost exclusively via correspondence. But, as we have described, the existence of the LMS fostered increased *sociabilités*, enabling face-to-face interactions between mathematicians, speeding up the exchange of mathematical ideas and making collaborations easier and quicker to undertake. Indeed, it is likely that Hardy and MacMahon's mutual membership and attendance at LMS meetings (and in particular on the LMS Council) led to their initial acquaintance and, ultimately, to MacMahon's offer of valuable help to Hardy.

But could it not be argued that, since both Hardy and MacMahon had connections with Cambridge, it is more likely that Cambridge was the real source of their collaboration rather than the LMS? Although MacMahon was made a member of St. John's College, Cambridge, in August 1904, according to his biographer Paul Garcia, "[f]rom that date, MacMahon maintained a loose association with St. John's College, which became stronger when he moved to Cambridge in 1922" [Garcia 2006, 79]. Indeed, during the entire period of the Hardy-Ramanujan collaboration, MacMahon continued to live and work in London [Garcia 2006, 137], while Hardy lived in Cambridge. Thus, although MacMahon certainly did visit Cambridge between 1904 and 1922, his visits there (and interactions with mathematicians like Hardy and Ramanujan) would not have been as common or as regular during those years as the meetings of the LMS, which took place every month.

Furthermore, since Hardy had been a member of the LMS since January 1901, and had been attending LMS meetings at the same time as MacMahon for three and a half years before MacMahon's association with Cambridge began, it is far more likely that Hardy and MacMahon first met, not at Trinity or St. John's College in Cambridge, but at a meeting of the LMS in London. For the same reason, it is also far more probable that the Hardy-MacMahon exchange was fostered by their mutual membership of the LMS (and regular attendance of LMS meetings) than by MacMahon's "loose association" with a Cambridge college.

We have said much on the exchange between Hardy and MacMahon, but what of that between MacMahon and Ramanujan? And to what extent did it involve the LMS? It may seem ironic, but it is interesting to note that in the entire period Ramanujan lived in Britain (1914-1919), he never attended a single LMS meeting. Nevertheless, he still managed to exert an influence on fellow LMS members, particularly MacMahon. Indeed, shortly after Ramanujan died, it was MacMahon—not Hardy or Littlewood—who presented a talk on the impact of Ramanujan's work at a meeting of the London Mathematical Society on 10 June 1920 [Garcia 2006, 134], [LMS Proceedings 1921, 19: xxvii]. Furthermore, not only was MacMahon's subsequent research on partitions directly influenced by Ramanujan's ideas, but his culminating paper on the parity of $p(n)$ arose directly from a question put to him by Ramanujan, and was published in the *Journal* of the LMS in 1926.

We thus see several forms of mathematical exchange at work within the context of the Hardy-Ramanujan-MacMahon-LMS collaboration. First, we see the exchange of western and non-western styles of doing mathematics, where Ramanujan's instinctive dexterity with formula-derivation met Hardy's genius for analytic proof. Secondly, within the context of the LMS, we see the exchange between amateur mathematicians like MacMahon and Ramanujan on the one hand, and Hardy representing the rising professional research mathematician on the other. We also see a dialogue between three quite different educational backgrounds: the Cambridge academic (Hardy), the self-taught mathematician (Ramanujan) and the military man (MacMahon). Fourthly, these backgrounds influenced the exchange of mathematical styles used in this collaboration, from the purely abstract and theoretical methods of Hardy & Ramanujan's proof to the brute-force computational techniques required for MacMahon's numerical data. These mathematical styles also represented a fusion of British and European mathematical methods, with Hardy and

Ramanujan incorporating their own work with material taken from French work on elliptic and modular functions and Scandinavian work on asymptotic series. Finally, by publishing in both the *Proceedings of the LMS* and in the *Comptes Rendus*, Hardy and Ramanujan chose to maximize the chances of further mathematical exchange by exposing their result to both a British and an international audience.

So, how much credit can be given to the LMS for the Hardy-Ramanujan formula? Given the sheer productivity and unbridled genius of Hardy and Ramanujan at this point in time, it is likely that their joint work on partitions would have occurred regardless of whether the LMS existed or not. But I would argue that it was the LMS, the venue of its initial presentation, which ensured the vital input of MacMahon, a rigorous refereeing process, and the place of its eventual publication. Without these crucial factors, the exact form of Hardy & Ramanujan's paper, the precision of the results it contained, and the influence it was able to exert would all have been very different. Thus while the London Mathematical Society may not have been an essential ingredient in this collaboration, it served as a useful catalyst for one of the most significant mathematical exchanges in early twentieth-century British mathematics.

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